

A Simple Continuous Parametrization of the Kasner Indices

Alex Harvey^{a)}

Visiting Scholar
New York University
New York, NY 10003

Abstract

A parametrization of the Kasner indices in terms of a continuous parameter is constructed by exploiting their representation as trilinear coordinates. This provides a clear picture of their variation through their entire range *vis a vis* each other. The parameter can be expressed as a function of time.

1 The Kasner Metric

The canonical form of the Kasner metric is

$$ds^2 = -dt^2 + t^{2a}dx^2 + t^{2b}dy^2 + t^{2c}dz^2 \quad (1)$$

where the indices $[a, b, c]$ must satisfy

$$a + b + c = 1 \quad (2a)$$

$$a^2 + b^2 + c^2 = 1 \quad (2b)$$

This is *not* Kasner's original formulation.¹

Equation (2a) implies that the Kasner indices may be treated as *trilinear coordinates* provided the *reference triangle* is equilateral.² If so treated, then Eq. (2b) is the locus of all points satisfying *both* Kasner conditions. This is readily found. If Eq. (2a) is squared and subtracted from Eq. (2b), the result is

$$ab + bc + ca = 0 \quad (3)$$

which, through division by abc , becomes

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0. \quad (4)$$

¹An exhaustive discussion of the Kasner metric was published by Harvey [1].

²See the Appendix.

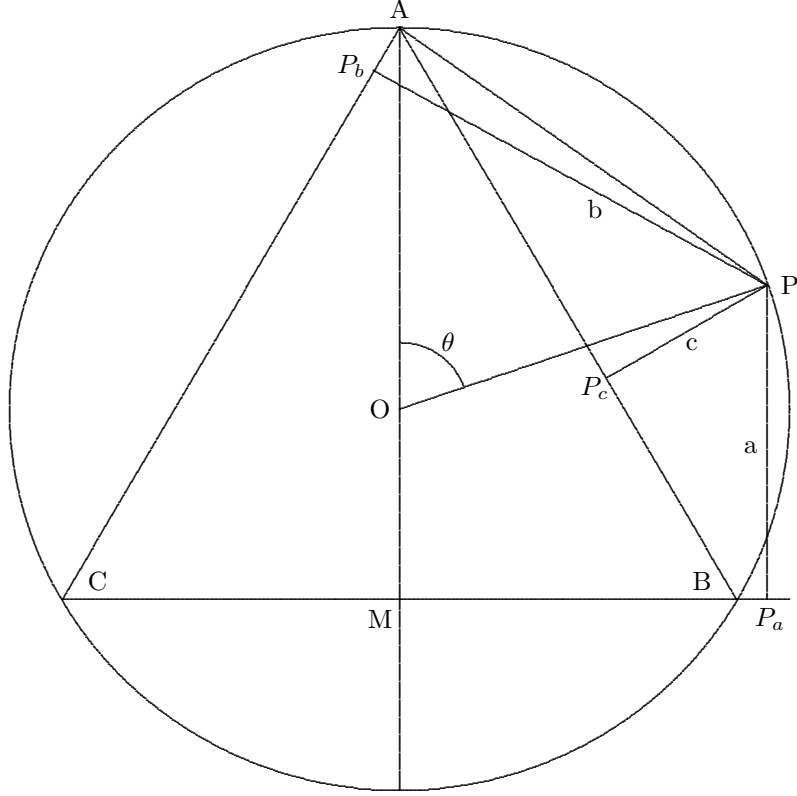


Figure 1: Kasner Coefficients in Trilinear Coordinates

This is precisely the equation of a circle in trilinear coordinates if the reference triangle is, as assumed, equilateral.

The linear Kasner condition implies that the altitude of the reference triangle must be equal to 1. The circle Eq. (4) will be its circumscribed circle which is thus the desired locus. A point P on the circle (see Fig. (1)) selects a set of values for $[a, b, c]$. As P moves (clockwise) from vertex A to B the possible values of the indices are obtained. As P progresses further from vertex B to C , the sets of values for $[a, b, c]$ become those for $[b, c, a]$ and similarly for P moving from C on to A .

The radius of the circle is $2/3$. The segment OM is $1/3$. Thus

$$a = \frac{1}{3} + \frac{2}{3} \cos(\theta). \quad (5)$$

Triangle AOP is isosceles with base

$$AP = 2 \left(\frac{2}{3} \sin \frac{\theta}{2} \right). \quad (6)$$

$$\begin{aligned}
\angle OAP &= \angle OPA = 90^\circ - \frac{\theta}{2} \quad \text{and} \\
\angle P_bAP &= 30^\circ + \angle OAP \quad \text{therefore} \\
\angle P_bAP &= 120^\circ - \frac{\theta}{2}
\end{aligned} \tag{7}$$

Similarly,

$$\angle P_cAP = 60^\circ - \frac{\theta}{2}. \tag{8}$$

The altitude of the reference triangle is 1 and the radius of the circumscribed circle is $2/3$. Consequently

$$\begin{aligned}
b &= AP \sin \left(120^\circ - \frac{\theta}{2} \right) \\
&= \frac{4}{3} \sin \frac{\theta}{2} \left(\frac{\sqrt{3}}{2} \cos \frac{\theta}{2} + \frac{1}{2} \sin \frac{\theta}{2} \right)
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
c &= -AP \sin \left(60^\circ - \frac{\theta}{2} \right) \\
&= -\frac{4}{3} \sin \frac{\theta}{2} \left(\frac{\sqrt{3}}{2} \cos \frac{\theta}{2} - \frac{1}{2} \sin \frac{\theta}{2} \right)
\end{aligned} \tag{10}$$

Figure (2) clearly shows how the values of the indices vary as $0 \leq \theta \leq 2\pi/3$. At the end points are the degenerate values, $[1, 0, 0]$; at least one of the indices must be negative; and midway between the endpoints are the special sets of values $[2/3, -1/3, 2/3]$. Figure (3) shows the interchange of index values over a complete rotation of OP over 2π . The parameter θ may be made a time modulated argument, that is $\theta(t)$. It may not be amplitude modulated as this would violate the linear Kasner condition.

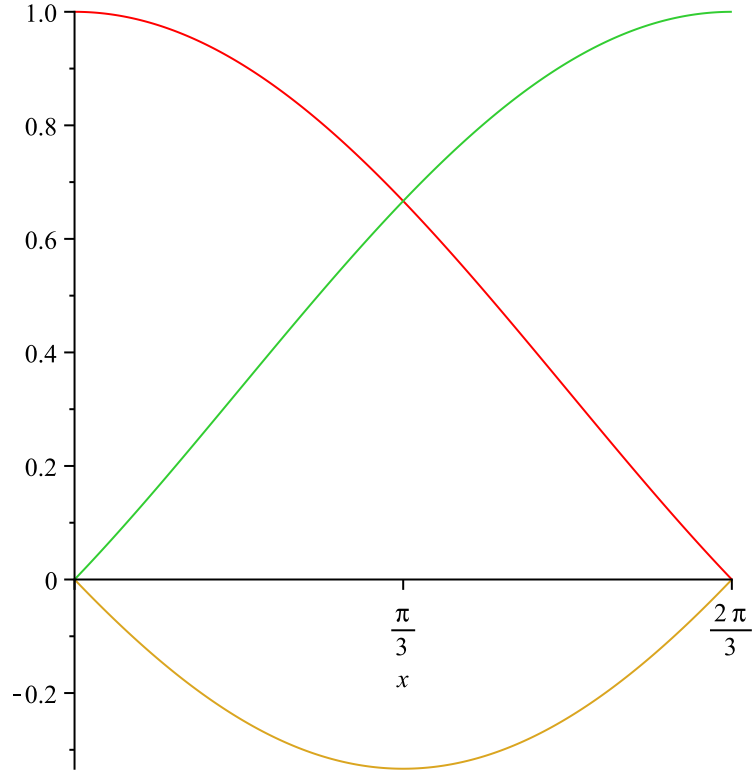


Figure 2: Numerical Range of Indices as P moves from A to B on the circle. The indices $[a, b, c]$ are identified by red, green, and yellow respectively.

A Trilinear Coordinates

Let 3 arbitrary, coplanar lines intersect to form a triangle $[A, B, C]$.³ If this triangle is designated the reference triangle then the trilinear coordinates of a point, P , are defined by

$$a\alpha + b\beta + c\gamma = 2\Delta \quad (11)$$

where $[\alpha, \beta, \gamma]$ are the perpendicular distances to the respective sides. (See Fig. (5).) It is obvious that Δ is the area of the triangle. The sign of each coordinate is positive or negative as the perpendiculars dropped from P onto the sides of the reference triangle terminate “inside” or “outside” the (extended) sides of the triangle. In Figure (5), the coordinates of P_1 are all positive. For P_2 , α_2 and β_2 are positive; γ_2 is negative.

³in this section, in conformity with the literature on plane geometry generally and trilinear coordinates [2] on particular, the set of symbols $[a, b, c]$ identifies the sides of the reference triangle.

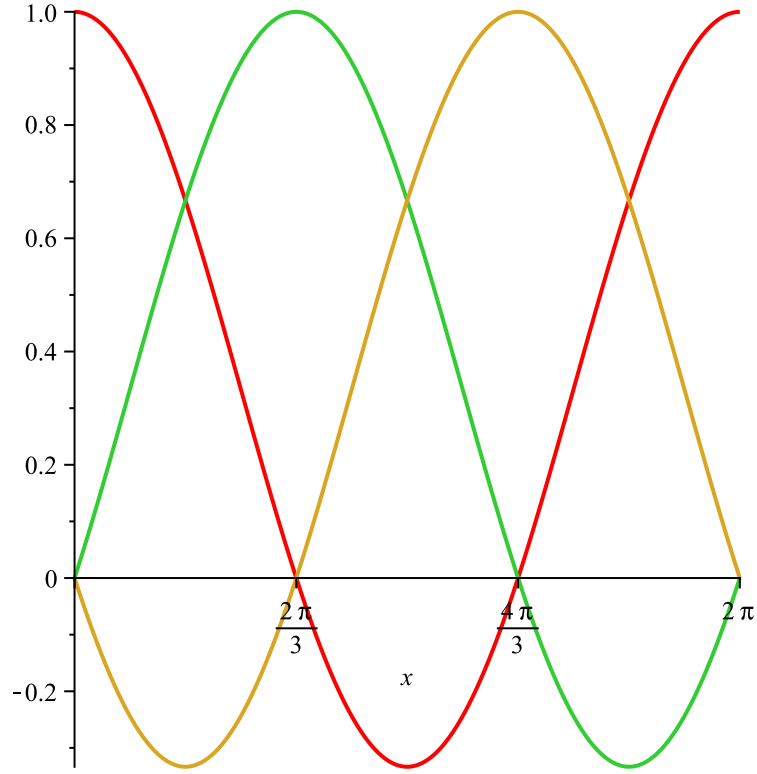


Figure 3: Complete Cycle of Indices as P moves through 2π .

If the reference triangle is equilateral with sides s then Eq. (11) reduces to

$$\alpha + \beta + \gamma = \frac{2\Delta}{s}. \quad (12)$$

The right side may be normalized to 1 to obtain the linear Kasner condition.

B Acknowledgment

That the Kasner conditions might be interpreted in terms of trilinear coordinates was suggested by Professor Engelbert Schucking. The diagrams were prepared by Dr. Eugene Surowitz. His work is deeply appreciated.

^{a)}Prof. Emeritus, Queens College, City University of New York, email: ah30@nyu.edu

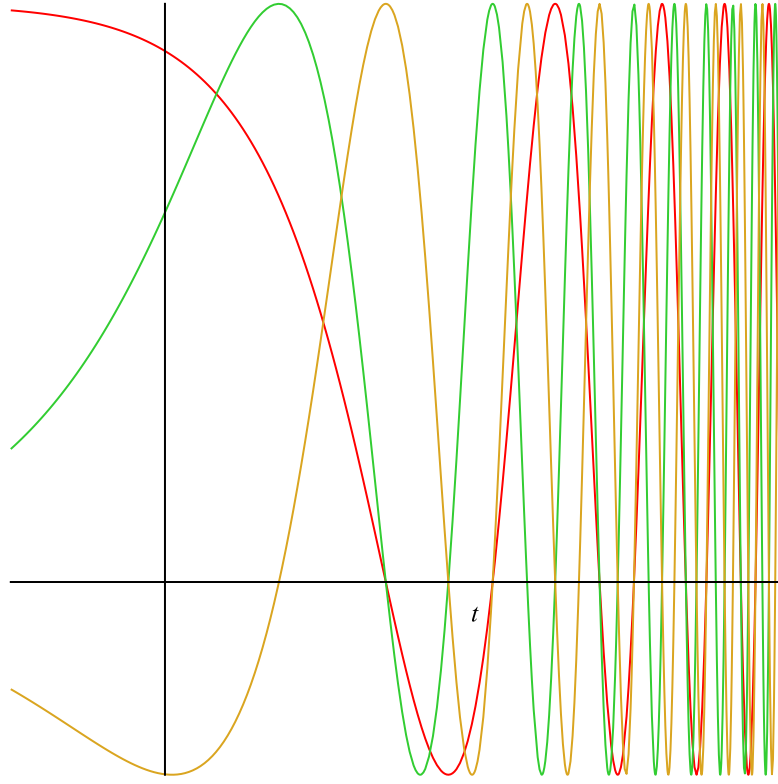


Figure 4: Variation of Index Values for Increasing Rate of Fluctuation

References

- [1] A Harvey, “Will the Real Kasner Metric Please Stand Up”, *General Relativity and Gravitation*, **22**, 1433-1435 (1990).
- [2] N M Ferrers, “An Elementary Treatise on Trilinear Coordinates”, 4th ed., Macmillan and Co., London (1890). This is available on line at: <http://www.archive.org/details/elementarytrili00ferruoft>. In accordance with the convention noted earlier, Ferrers designates the sides of the reference triangle $[a, b, c]$, and the coordinates of a point by $[\alpha, \beta, \gamma]$.

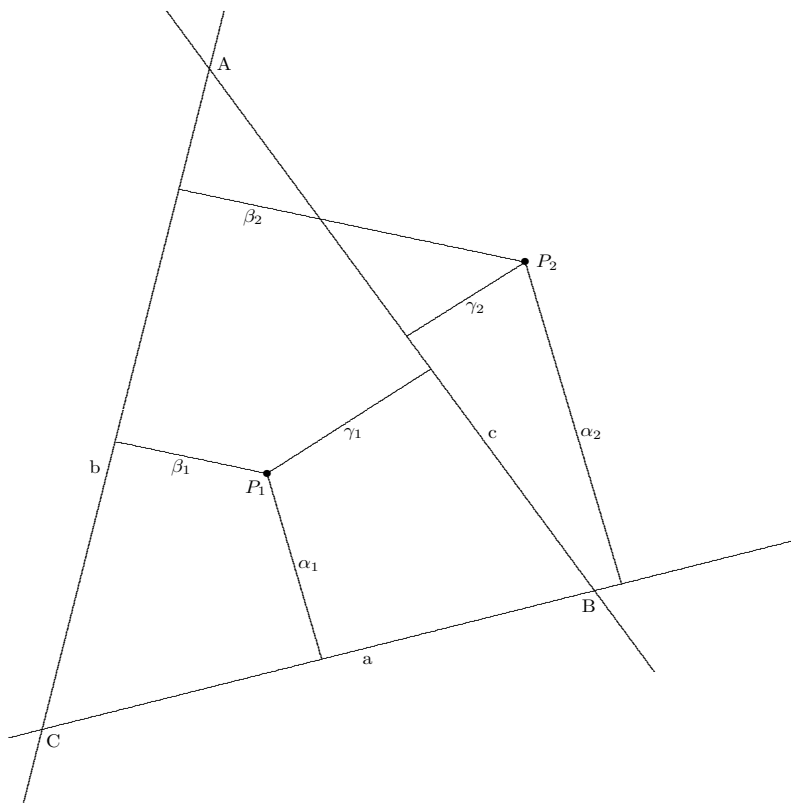


Figure 5: Arbitrary Reference Triangle